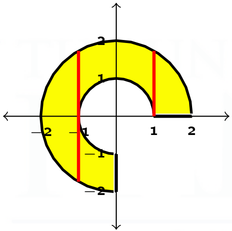
Section 15.6
Change of Variables

PreLecture Video

1 Transformations and Jacobians

Motivating Example

Motivating Example: Calculate $\iint_D dA$, where D is the region shown below. ‣ Motivating Video



- Approach 1: Break D into simple regions or use two concentric regions. This will involve Square root functions. (Yuck.)
- Approach 2: Use polar coordinates.
  - Observe that $D = \{(r,\theta) \mid 1 \leq r \leq 2,\ 0 \leq \theta \leq \frac{3\pi}{2}\}$.
  - Goal: Write $\iint_D dA$ as $\int_1^2 \int_0^{3\pi/2} [\text{something}]\, d\theta\, dr$

Change of Variables

In general, we want to be able to evaluate integrals $\iint_D f(x,y)\,dA$, where D has a complicated shape, by replacing D with a simpler region E. The relationship between the regions is given by a transformation $G : E \to D$.

The punchline will be that

$$\iint_D f(x,y)\,dA = \iint_E f(G(u,v))\,|\text{Jac}(G)|\,du\,dv$$

where $\text{Jac}(G)$, the **Jacobian** of G, records how G rescales area.

Things we need to figure out:

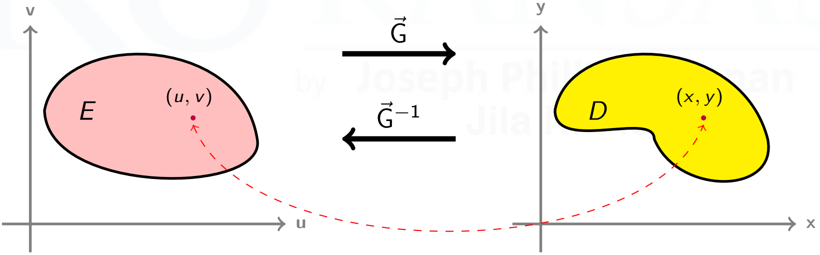
1. What is a "transformation"?
2. How do we measure how a transformation rescales area?

Transformations in $\mathbb{R}^2$

Suppose that we have planar regions D (with coordinates x, y) and E (with coordinates u, v).

A **transformation** from E to D is a function $\vec{G}(u, v) = (x, y)$ such that

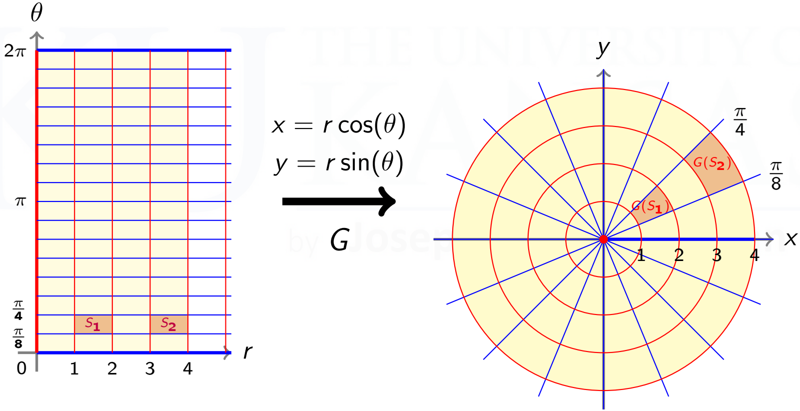
1. $\vec{G}$ is one-to-one on the interior of E.
2. $\vec{G}$ has continuous first-order partial derivatives.



Typically, transformations change the area of regions in $\mathbb{R}^2$.

Transformations in $\mathbb{R}^2$

Motivating Example: Conversion of rectangular coordinates to polar coordinates is a transformation $[0, \infty) \times [0, 2\pi] \to \mathbb{R}^2$.



Link

Transformations in $\mathbb{R}^2$

To see that $\vec{G}(r,\theta) = (x,y) = (r\cos(\theta), r\sin(\theta))$ is a transformation:

1. $\vec{G}$ is **one-to-one** on $(0,\infty)\times(0,2\pi)$.

(It is not invertible on the boundary — for example, $\vec{G}(r,0) = \vec{G}(r,2\pi)$ for all r, and $\vec{G}(0,\theta) = (0,0)$ for all θ – but that is okay.)

2. $\vec{G}$ is **continuously differentiable**:

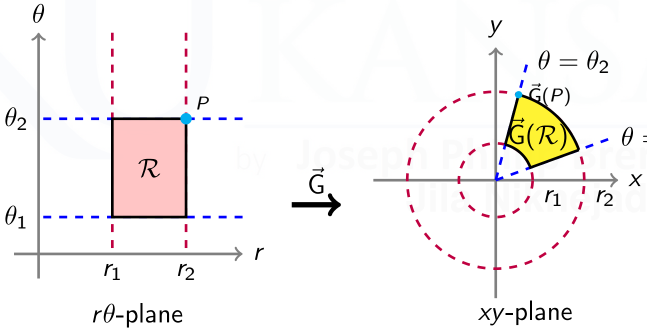
$$x_r(r,\theta) = \cos(\theta) \qquad x_\theta(r,\theta) = -r\sin(\theta)$$
$$y_r(r,\theta) = \sin(\theta) \qquad y_\theta(r,\theta) = r\cos(\theta)$$

$$\vec{G}_r(r,\theta) = (\underbrace{\cos(\theta)}_{x_r}, \underbrace{\sin(\theta)}_{y_r}) \qquad \vec{G}_\theta(r,\theta) = (\underbrace{-r\sin(\theta)}_{x_\theta}, \underbrace{r\cos(\theta)}_{y_\theta})$$

Transformations in $\mathbb{R}^2$

A transformation $\vec{G} : E \to D$ doesn't just map points to points; it maps subsets A of the domain E to subsets of the range D.

For instance, if $\vec{G}(r, \theta) = (x, y) = (r\cos(\theta), r\sin(\theta))$ and $\mathcal{R}$ is the rectangle $[r_1, r_2] \times [\theta_1, \theta_2]$, then $\vec{G}(\mathcal{R})$ is a circular sector:

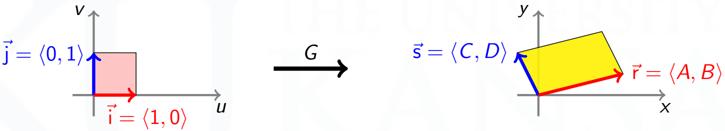


Note: The area of $\vec{G}(\mathcal{R})$ does **not** depend just on the area of $\mathcal{R}$!

Linear Transformations

The simplest transformations are **linear transformations**. (MATH 290!)

$\vec{G}(u, v) = \big(Au + Cv, Bu + Dv\big)$ (where A, B, C, D are constants)



Let $\vec{r} = \langle A, B\rangle$ and $\vec{s} = \langle C, D\rangle$. Then:

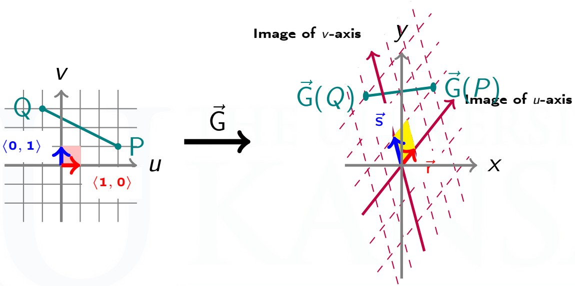
- $\vec{r} = \vec{G}(\vec{i})$ and $\vec{s} = \vec{G}(\vec{j})$.
- G transforms the unit square $[0, 1] \times [0, 1]$ to a parallelogram with sides $\vec{r}$ and $\vec{s}$.
- The unit square has area $\|\vec{i} \times \vec{j}\| = 1$, and the parallelogram has area $\|\vec{r} \times \vec{s}\| = |AD - BC|$. *(This is the rescaling factor for all rectangular regions.)*

In Math 290, the above linear transformation is denoted as

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} A & C \\ B & D \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}$$

Linear Transformations

G maps translations of $[0,1]\times[0,1]$ to translations of the parallelogram.



Linear transformations scale area uniformly. The scaling factor is the absolute value of the determinant

$$\det\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{vmatrix} A & B \\ C & D \end{vmatrix} = AD - BC.$$

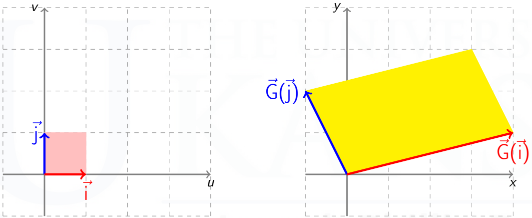
That is, for all regions E in the u,v-plane,

$$\text{Area}\big(\vec{G}(E)\big) = |AD - BC|\,\text{Area}(E).$$

Linear Transformations

Example 1:

$\vec{G}(u,v) = (\underbrace{4u - v}_{x}, \underbrace{u + 2v}_{y})$.



The scaling factor for area is $\begin{vmatrix} 4 & 1 \\ -1 & 2 \end{vmatrix} = 9.$

The inverse of $\vec{G}$ is $\vec{G}^{-1}(x,y) = \left(\dfrac{y+2x}{9}, \dfrac{4y-x}{9}\right)$.

(Note: If $\vec{G} : \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation and the area scaling factor is nonzero, then $\vec{G}$ is invertible.)

2 Integration and Change of Variables

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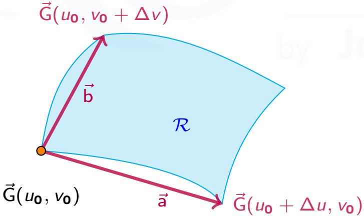
Rescaling Area for General Transformations

What is the scaling factor for a general (non-linear) transformation?

Take a **very small** rectangle E in the uv-plane with a vertex at (u_0, v_0) and side lengths Δu and Δv.

Suppose that E is mapped to R in the xy-plane by a transformation G.



The region R is not necessarily a rectangle, but it does have four vertices and four edges.

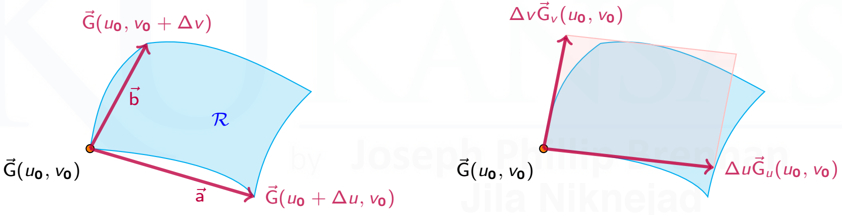
Since the rectangle E was very small, the edges connected to $G(u_0, v_0)$ can be approximated by the vectors

$$\vec{a} \approx \Delta u \left\langle \frac{\partial x}{\partial u}, \frac{\partial y}{\partial u} \right\rangle = \Delta u\, \vec{G}_u \qquad \vec{b} \approx \Delta v \left\langle \frac{\partial x}{\partial v}, \frac{\partial y}{\partial v} \right\rangle = \Delta v\, \vec{G}_v$$

Rescaling Area for General Transformations

Since the rectangle E was *very small*, its image $R = \vec{G}(E)$ is close to a parallelogram, so

$$\text{area}(R) \approx \left\| \left(\vec{G}_u \, \Delta u\right) \times \left(\vec{G}_v \, \Delta v\right) \right\| = \left\| \vec{G}_u \times \vec{G}_v \right\| \Delta u \, \Delta v$$



Conclusion: The transformation G scales area by a factor of

$$\left\| \vec{G}_u \times \vec{G}_v \right\| .$$

Jacobians and the Change-Of-Variable Formula

The **Jacobian** of the transformation $\vec{G}(u,v) = (x(u,v), y(u,v))$ is defined as

$$\left|\text{Jac}(\vec{G})\right| = \left|\frac{\partial(x,y)}{\partial(u,v)}\right| = \|\vec{G}_u \times \vec{G}_v\| = \left\|\begin{matrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{matrix}\right\| = \left|\frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}\right|$$

The absolute value of Jacobian is the area scaling factor for $\vec{G}$. That is, the scaling factor is $\|\vec{G}_u \times \vec{G}_v\|$.

Double Integration with Change of Variables

Let $\vec{G}(u,v) = (x(u,v), y(u,v))$ be a transformation, and let $\vec{G}(S) = R$. Then

$$\iint_R f(x,y)\, dA_{xy} = \iint_S f(x(u,v), y(u,v)) \left|\frac{\partial(x,y)}{\partial(u,v)}\right| dA_{uv}$$

Using this formula does not typically evaluate the integral immediately, but it enables you to convert it into an integral over a geometrically simpler region or simpler integrand.

Interlude: Change of Variables vs. u-Substitution

The change-of-variables formula

$$\iint_R f(x,y)\,dA_{xy} = \iint_S f(x(u,v),y(u,v))\left|\frac{\partial(x,y)}{\partial(u,v)}\right|\,dA_{uv}$$

is analogous to u-substitution from Calculus I:

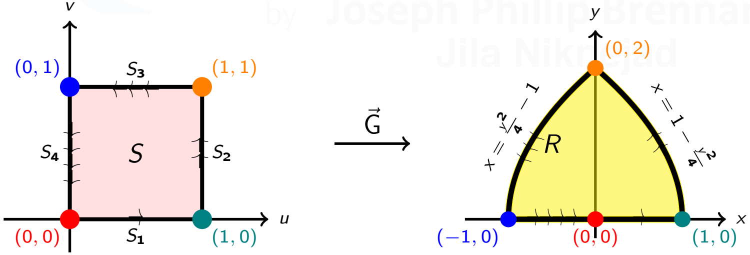
$$\int_a^b F(g(x))g'(x)\,dx \;=\; \int_{g(a)}^{g(b)} F(u)\,du$$

- In Calculus I, we used a transformation $u = g(x)$ to replace an integral over x with an integral over u, in order to simplify the **integrand**.
- Now, we are using a transformation $(x,y) = G(u,v)$ to replace an integral over x,y with an integral over u,v, either to simplify the integrand or the **region of integration**.

Example 2: Let $\vec{G}$ be the transformation given by $x = u^2 - v^2$ and $y = 2uv$ and let $S = \{(u,v) \,|\, 0 \le u \le 1\,,\, 0 \le v \le 1\}$. Find the image $R = \vec{G}(S)$, the Jacobian of G, and the area of R.

Solution: Walk around the boundary of S and plot the corresponding points on the boundary of $\vec{G}(S)$. In this case, the change of variable simplifies the region.

S	$G(S)$	R	
$(u,0),\ 0 \le u \le 1$	$(u^2, 0)$	$y = 0$	$0 \le x \le 1$
$(1,v),\ 0 \le v \le 1$	$(1 - v^2, 2v)$	$x = 1 - y^2/4$	$0 \le y \le 2$
$(u,1),\ 1 \ge u \ge 0$	$(u^2 - 1, 2u)$	$x = y^2/4 - 1$	$2 \ge y \ge 0$
$(0,v),\ 1 \ge v \ge 0$	$(-v^2, 0)$	$y = 0$	$-1 \le x \le 0$



Example 2 (cont'd):

$$\vec{G}(u,v) = (x,y) = (\underbrace{u^2 - v^2}_{x},\ \underbrace{2uv}_{y})$$

$$S = \{(u,v) \mid 0 \le u \le 1\,,\ 0 \le v \le 1\}$$

The Jacobian is $\mathrm{Jac}(\vec{G}) = \dfrac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} 2u & -2v \\ 2v & 2u \end{vmatrix} = 4(u^2+v^2).$

We can calculate the area of $R = \vec{G}(S)$ in two ways.

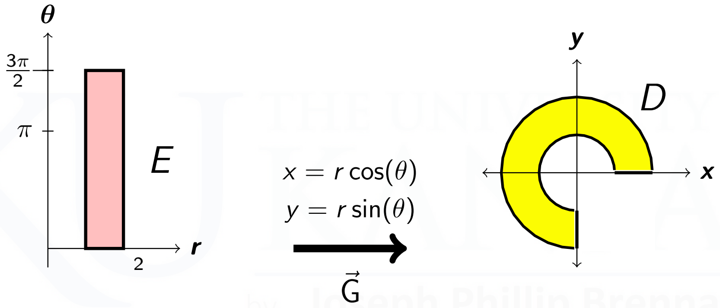
Using the change-of-variables formula:

$$\iint_R 1\,dA = \iint_S 4(u^2+v^2)\,dA = 4\int_0^1\int_0^1 (u^2+v^2)\,du\,dv = \frac{8}{3}$$

Using single-variable calculus:

$$2\int_0^2 1 - \frac{y^2}{4}\,dy \;=\; 2\left(y - \frac{y^3}{12}\bigg|_0^2\right) = \frac{8}{3}$$

Motivating Example Revisited (for handout only): Calculate $\iint_D x^2y\,dA$, where D is the region shown below.



Here we can use change of variables to simplify the domain of integration by replacing D with E.

By the change-of-variables formula:

$$\iint_D x^2y\,dA_{xy} = \iint_E (r\cos\theta)^2(r\sin\theta)\begin{vmatrix} x_r & x_\theta \\ y_r & y_\theta \end{vmatrix} dA_{r\theta}$$

Motivating Example Revisited (cont'd):

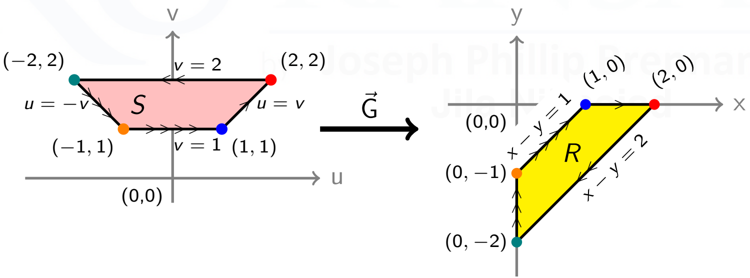
$$
\begin{aligned}
\iint_D x^2 y \, dA_{xy} &= \iint_E (r\cos\theta)^2 (r\sin\theta) \begin{vmatrix} x_r & x_\theta \\ y_r & y_\theta \end{vmatrix} dA_{r\theta} \\
&= \int_1^2 \int_0^{3\pi/2} (r\cos(\theta))^2 (r\sin(\theta)) \begin{vmatrix} \cos(\theta) & -r\sin(\theta) \\ \sin(\theta) & r\cos(\theta) \end{vmatrix} d\theta \, dr \\
&= \int_1^2 \int_0^{3\pi/2} r^3 \cos^2(\theta)\sin(\theta) \left(r\cos^2(\theta) + r\sin^2(\theta) \right) d\theta \, dr \\
&= \int_1^2 \int_0^{3\pi/2} r^4 \cos^2(\theta)\sin(\theta) d\theta \, dr \\
&= \left(\int_1^2 r^4 \, dr \right) \left(\int_0^{3\pi/2} \cos^2(\theta)\sin(\theta) d\theta \right) \\
&= \left(\frac{31}{5} \right) \left(-\frac{1}{3} \right) = -\frac{31}{15}.
\end{aligned}
$$

Example 3: Let R be the trapezoidal region with vertices $(1,0)$, $(2,0)$, $(0,-2)$, and $(0,-1)$. Evaluate the integral

$$\iint_{\mathcal{R}} e^{(x+y)/(x-y)}\, dA.$$

Solution: Here we can use change of variables to simplify the integrand.

- The integrand suggests defining $(u,v) = \vec{G}^{-1}(x,y) = (x+y,\ x-y)$.
- Solve for x, y to get $(x,y) = \vec{G}(u,v) = \left(\dfrac{u+v}{2}, \dfrac{u-v}{2}\right)$.



Note that S is horizontally simple: $1 \le v \le 2$, $-v \le u \le v$.

Example 3 (cont'd):

$$\begin{aligned}
\iint_R e^{(x+y)/(x-y)}\,dA &= \iint_S e^{u/v}\left|\frac{\partial(x,y)}{\partial(u,v)}\right| dA \\
&= \int_1^2 \int_{-v}^{v} e^{u/v} \begin{vmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{vmatrix}\, du\, dv \\
&= \frac{1}{2}\int_1^2 \int_{-v}^{v} e^{u/v}\, du\, dv \\
&= \frac{1}{2}\int_1^2 \left[ve^{u/v}\Big|_{u=-v}^{u=v} \right] dv \\
&= \frac{1}{2}\int_1^2 v(e-e^{-1})\, dv \\
&= \frac{3}{4}\left(e-e^{-1}\right)
\end{aligned}$$

A Useful Fact About Jacobians

If F is the inverse transformation of G, that is,

$$\vec{\mathsf{F}}(x,y) = (u,v) \quad \text{and} \quad \vec{\mathsf{G}}(u,v) = (x,y),$$

then

$$\mathrm{Jac}(\vec{\mathsf{F}}) = \mathrm{Jac}(\vec{\mathsf{G}})^{-1}.$$

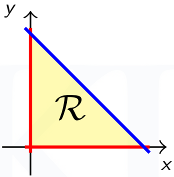
This fact is suggested by the notation:

$$\mathrm{Jac}(\vec{\mathsf{F}}) = \frac{\partial(u,v)}{\partial(x,y)}, \qquad \mathrm{Jac}(\vec{\mathsf{G}}) = \frac{\partial(x,y)}{\partial(u,v)}$$

(Try it yourself for a linear transformation — or see exercises 49-51 in §15.6.)

Example 4: Evaluate $\displaystyle\int_0^1 \int_0^{1-x} \sqrt{x+y}\,(y-2x)^2\,dy\,dx$

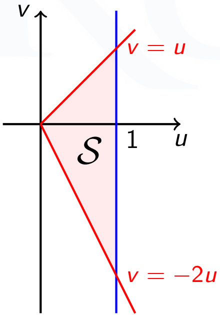
<u>Solution:</u>



The domain is simple, so we use the transformation

$$\vec{G}^{-1}(x,y) = (x+y,\, y-2x)$$

to simplify the integrand and hope that the new domain is still simple!



$$\frac{\partial(u,v)}{\partial(x,y)} = 3 \quad \Rightarrow \quad \frac{\partial(x,y)}{\partial(u,v)} = \frac{1}{3}$$

$$\iint_{\mathcal{R}} \sqrt{x+y}(y-2x)^2\,dA = \int_0^1 \int_{-2u}^{u} \frac{\sqrt{u}v^2}{3}\,dv\,du$$

$$= \int_0^1 u^{7/2}\,du = \frac{2}{9}$$

Change of Variables, Simplifying the Domain

Example 5: Let $\mathcal{R}$ be the region in the first quadrant bounded by $xy = 1$, $xy = 4$, $y = x$, and $y = 2x$. Evaluate the integral

$$\iint_{\mathcal{R}} xy^3 \, dA.$$

<u>Solution:</u> The domain of integration is

$$\mathcal{R} = \left\{(x, y) \;\middle|\; 1 \le xy \le 4,\ 1 \le y/x \le 2\right\}.$$

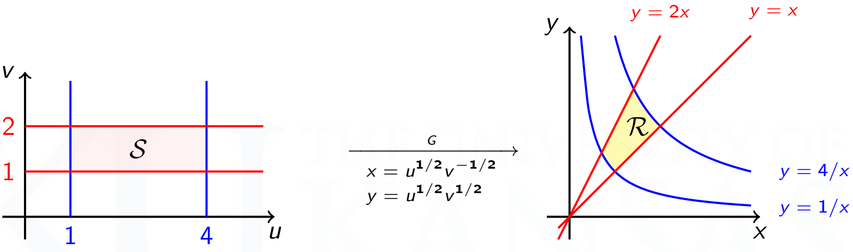
Wouldn't it be nice if xy and y/x were variables so that $\mathcal{R}$ was a rectangle?

Use a transformation! Define

$$\vec{\mathrm{G}}^{-1}(x, y) = (u, v) = (xy,\ y/x)$$

so that

$$\vec{\mathrm{G}}(u, v) = (x, y) = (u^{1/2}v^{-1/2},\ u^{1/2}v^{1/2}).$$



Jacobian: $$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{1}{2}u^{-1/2}v^{-1/2} & -\frac{1}{2}u^{1/2}v^{-3/2} \\ \frac{1}{2}u^{-1/2}v^{1/2} & \frac{1}{2}u^{1/2}v^{-1/2} \end{vmatrix} = \frac{1}{2v}$$

In general: $$\iint_{\mathcal{R}} f(x,y)\,dA = \int_1^4 \int_1^2 f\left(u^{1/2}v^{-1/2},\ u^{1/2}v^{1/2}\right) \frac{1}{2v}\,dv\,du$$

In particular: $$\iint_{\mathcal{R}} xy^3\,dA = \int_1^4 \int_1^2 \frac{u^2}{2}\,dv\,du = \frac{21}{2}.$$

3 Change of Variables for Triple Integrals

Change of Variables for Triple Integrals

Let R be a region in $\mathbb{R}^3$ with coordinates x, y, z.
Let S be a region in $\mathbb{R}^3$ with coordinates u, v, w.
Let $\vec{\mathrm{G}}$ be a transformation that maps S to R:

$$\vec{\mathrm{G}}(u, v, w) = \big(x(u, v, w),\ y(u, v, w),\ z(u, v, w)\big).$$

Then

$$\iiint_R f(x, y, z)\, dV_{xyz} = \iiint_S f\big(G(u, v, w)\big) \left| \frac{\partial(x, y, z)}{\partial(u, v, w)} \right| dV_{uvw}$$

where

$$\frac{\partial(x, y, z)}{\partial(u, v, w)} = \mathrm{Jac}(\vec{\mathrm{G}}) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}.$$

Change of Variables for Triple Integrals Example

Example 6: Let R be the parallelepiped in $\mathbb{R}^3$ defined by the inequalities

$$\begin{array}{ccccc} 1 & \leq & x-2y+z & \leq & 3, \\ 2 & \leq & 2x+y-3z & \leq & 4, \\ 5 & \leq & x+y+z & \leq & 8. \end{array}$$

To change variables in an integral of the form $\iiint_R f(x,y,z)\,dV$:

- Let $(u,v,w) = \vec{G}^{-1}(x,y,z) = (x-2y+z,\ 2x+y-3z,\ x+y+z)$.

  In Math 290, this is also known as $\vec{G}(x,y,z) = \begin{bmatrix} 1 & -2 & 1 \\ 2 & 1 & -3 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$.

- The inverse is $(x,y,z) = \vec{G}(u,v,w) = \left(\frac{4u+3v+5w}{15}, \frac{-5u+5w}{15}, \frac{u-3v+5w}{15}\right)$.

  In Math 290, this is known as $\begin{bmatrix} 1 & -2 & 1 \\ 2 & 1 & -3 \\ 1 & 1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} u \\ v \\ w \end{bmatrix}$

- Compute $\text{Jac}(\vec{G}) = \text{Jac}(\vec{F})^{-1} = \begin{vmatrix} 1 & -2 & 1 \\ 2 & 1 & -3 \\ 1 & 1 & 1 \end{vmatrix}^{-1} = \frac{1}{15}$.

Change of Variables for Triple Integrals Example

Example 6 (cont'd): The upshot is that

$$\iiint_R f(x, y, z)\, dV_{xyz} =$$

$$\int_1^3 \int_2^4 \int_5^8 f\left(\frac{4u + 3v + 5w}{15}, \frac{-5u + 5w}{15}, \frac{u - 3v + 5w}{15}\right) \frac{1}{15}\, dw\, dv\, du.$$

We wanted to present a general example of linear change of variables for triple integral so you connect math 127 and Math 290 transformations. You will see a simple example of linear change of variables for triple integral in lab which doesn't require any knowledge of Math 290.

The most common applications of the change-of-variables formulas are to convert double integrals to polar coordinates, and to convert triple integrals to cylindrical or spherical coordinates.